A family of regula falsi root-finding methods

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***Abstract*—A family of regula falsi methods for finding a root ξ of a nonlinear equation f(x)=0 in the interval [a,b] is studied in this paper. The Illinois, Pegasus, Anderson & Björk and more nine new proposed methods have been tested on a series of examples. The new methods, inspired on Pegasus procedure, are pedagogically important algorithms. The numerical results empirically show that some of new methods are very effective.**

***Index Terms*—regula ralsi, root finding, nonlinear equations.**

I. Introduction

Is not so hard the design of heuristic procedures to solve a problem using numerical algorithms. The main drawback of this attitude is that you can be rediscovering the wheel. The famous Halley's iteration method [1] [1], have the distinction of being the most frequently rediscovered iterative method [2]. The Halley iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 - \frac{f(x_n)f''(x_n)}{2\cdot f'(x_n)^2}\right)^{-1}$$

The term in brackets is the correction of the Newton-Raphson method. Halley's method will yield cubic convergence at simple zeros of $f(x)$.

An attempt to create an useful numerical algorithm is try to improve the characteristics of already existent algorithm. The methods of Illinois [3] and Pegasus [4] are typical samples of improvements for the *Regula Falsi* method (false-position method) in order to solve fixed point retention problem. This is because once an interval has been reached on which the $f(x)$ is (monotonically) increasing or decreasing, one of end points is always retained. Xinyuan Wu, Zuhe Shen and Jianlin Xia [5] stated have developed a new root finding with global convergence for enclosing simple zeros x^* of nonlinear equation, which improves regula falsi method such that both the sequence of diameters $(b_n - a_n)$ and the iterative sequence $(x_n - x^*)$ are quadratically convergent to zero. Jinhai Chen and Weiguo Li [6], [7] give a class of regula falsi iterative formulas for solving nonlinear equations where both the sequences of diameters and iterative points sequence are quadratically convergent to zero. A powerful variant on false position is due to Ridders [8] has some very nice properties. First, is guaranteed to the iteration formula method never jumps out of its brackets. Second, the convergence is quadratic. Since each iteration requires two function evaluations, the actual order of the method is $\sqrt{2}$, not 2; but this is still quite respectably superlinear: The number of significant digits in the answer approximately doubles with each two function evaluations. Third, taking out the function's "bend" via exponential (that is, ratio) factors, rather than via a polynomial technique (e.g., fitting a parabola), turns out to give an extraordinarily robust algorithm. In both reliability and speed, Ridders' method is generally competitive with the more highly developed and better established (but more complicated) method of van Wijngaarden, Dekker, and Brent [9].

II. Method of Regula Falsi

Interval search procedure (or bracketing methods, are methods starting with two values of x which bracket the root, $x = \xi$, and systematically reduce the interval while keeping the root trapped within the interval) is a family of several methods for obtaining an approximate solution to an equation $f(x) = 0$. With Regula Falsi (false position) we begin with an interval $[a,b]$ in which $f(x)$ change its sign ($f(a) * f(b) < 0$). The linear interpolation is then used to find an approximate value of the true root ξ. We have

$$x = \frac{a\cdot f(b) - b\cdot f(a)}{f(b) - f(a)}$$

We can also derive the above equation considering the weighted mean (taking $|f(a)|$, $|f(b)|$ as weights).

We follows determining the subinterval $[a,x]$ or $[x,b]$, which contain the true root ξ by checking the sign of $f(x)$. If $f(x)\cdot f(a) < 0$ the $[a,x]$ contain the root, and x becomes the new $b]$ for the next iteration (as all iterative method, a sequence of approximations, $x_1, x_2 \cdots, x_n$, is generated to get closer to the true root). On the other hand, if $f(x)\cdot f(b) < 0$, $[x,b]$ contain the root, and x becomes the new $[a$ for the next iteration. The iterative process is repeated until one stopping criterion is attained. For example, the interaction is continued either one or both of the convergence criteria are satisfied

$$|f(x)| < \varepsilon_1 \; or |x_{n+1} - x_n| < \varepsilon$$

where are preassigned small positive numbers (related precision).

Regula Falsi sometimes has lower order de convergence. This is because once an interval has been reach on which the

[1]Edmund Halley (1656-1742), an English astronomer. Halley's Comet or Comet Halley (officially designated 1P/Halley) was only recognized as a periodic comet in the 18th century when its orbit was computed by Edmond Halley, after whom it is named , is the most famous of the periodic comets, and is visible from Earth every 75 to 76 years.

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f(x) is convex or concave, one of the end points is always retained. The method is often superlinear, but estimation of its exact order is not so easy [9].

III. A FAMILY OF ITERATIVE FORMULAS

If the function $f(x)$ is continuous in the interval $[a,b]$ and if $f(a)$ and $f(b)$ have opposite signs, then at least one root f(x) lie in interval $[a,b]$ (the *intermediate value theorem*). The family of methods generates a sequence x_i that converges to the *bracketed* root ξ of the equation $f(x)=0$. In the family of methods, based on linear approximations, the sequence x_i is obtained through the recurrence formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f(x_k)-f(x_k-1)} \cdot (x_k - x_{k-1}), \qquad k = 1,2,3,\ldots \tag{1}$$

The $[x_{k-1}, f(x_{k-1})]$ and $[x_k, f(x_k)]$ points are chosen so that $f(x_{k-1})$ e $f(x_k)$ always have opposed signs, guaranteed $\xi \in [x_{k-1}, x_k]$. The calculated value $f(x_{k-1})$ it is reduced by the factor $\phi[f(x_{k-1}), f(x_k), f(x_{k+1})]$, see Table I $[f(x_{k-1}), f(x_k), f(x_{k+1})] \longrightarrow [fa, fb, fx]$, in order to avoid the retention of a point as it happens in the it *regula falsi* method. In this way the straight line is drawn by a point no belonging the curve of $f(x)$. Should be mentioned the correspondences: Φ_1 to Pegasus, Φ_8 Illinois, and Φ_{12} Anderson & Björk. All others nine Φ_n corresponding to new proposed methods.

TABLE I
REDUCTION FACTOR FOR EACH FAMILY MEMBER

n	$\phi_{\mathbf{n}}(fa, fb, fx)$
1	$\dfrac{fa \cdot fb}{fb + fx}$
2	$\dfrac{fa - fb}{2}$
3	$\dfrac{fa - fx}{2 + fx/fb}$
4	$\dfrac{fa - fx}{(1 + fx/fb)^2}$
5	$\dfrac{fa - fx}{(1.5 + fx/fb)^2}$
6	$\dfrac{fa - fx}{(2 + fx/fb)^2}$
7	$\dfrac{fa + fx}{(2 + fx/fb)^2}$
8	$\dfrac{fa}{2}$
9	$\dfrac{fa}{(1 + fx/fb)^2}$
10	$\dfrac{fa - fx}{4}$
11	$\dfrac{fx \cdot fa}{fb + fx}$
12	$fa \cdot m,$ $m = \left(1 - \frac{fx}{fb}\right)$ $if\ positive$ $else\ m = 1/2$

The algorithms are variants of the regula falsi method in which given a new estimate, the estimate which is replaced is the one for which the sign of the function is the same as the sign of the function for the new estimate and the ordinate associated with the other estimate is reduced. The algorithm's family is guaranteed to converge.

IV. CASE STUDIES

The following Scilab script implement the algorithms of regula falsi family. It includes some basic practicalities. First, we enter the function of nonlinear equation $f(x) = 0$ (in line 2). Second, in line 11, the tolerance, maximum number of iterates and $f(x) = 0$ "root exact" are used for loop exit. Third, in line 17 we write the reduction factor Φ_n for each family member (see Table I). Finally, call the procedure (line 28) and show the results: root, number of iterates and error code (line 29).

```
1.  function y=f1(x)
2.  y=(x+2)*(x+1)*(x-3)^3    // from Tables 2 and 4 ( f1 from Table 4 case)
3.  endfunction
4.
5.  function [x, It, Erro] = Pegasus_1(f, a, b,  tol, ItMax)
6.  fa=f(a);fb=f(b);
7.  x=b;fx=fb;It=0;
8.  while %T
9.    It=It+1;dx=-fx/(fb-fa)*(b-a);
10.   x=x+dx;fx=f(x);
11.   if abs(dx) < tol |  It > ItMax | fx==0 then
12.     break
13.   end
14.   if fx*fb <0 then
15.     a=b;fa=fb;
16.   else
17.     fa=fa*fb/(fb+fx)      // from Table 1 ( Φ1 case )
18.   end
19.   b=x; fb=fx;
20.  end
21. if It > ItMax then
22.   Erro=1
23. else
24. Erro=0
25. end
26. endfunction
27.
28. [x, Iter, Error]= Pegasus_1( f1, 2.5 ,3.5, 1e-15,500);
29. [x, Iter, Error]
```

A. *Case 1*

Here we present the first numerical experiment. We tested all the problems listed in Table II. These nonlinear equations are well known (simple zeros). They are a series of published examples (see [5], [10], [11], [4], [3], etc.). Since the structures of the algorithms are the same, the number of iterations is used in the comparison. The Table III we list the total number of iterations for each individual problem. The factors of reduction Φ_{12}, Φ_4, Φ_1, and Φ_9 supplies the smallest total numbers of iterations, respectively.

B. *Case 2*

Now we present the second numerical experiment considering multiple roots (see Appendix). We tested all equations listed in Table IV with multiple roots and all its roots are real.

The family Regula falsi methods may not always work with satisfactory convergence speed. The performance will depend,

TABLE II

FIRST SET OF FUNCTIONS USED IN NUMERICAL EXPERIMENTS OF ROOT-FIND METHODS

k	$g_k(x)$	[a,b][1]
1	x^3-1	$[0.5,\ 1.5]$
2	$x^2(x^2/3+\sqrt{2}\cdot sin(x))-\sqrt{3}/18$	$[0.1,\ 1]$
3	$11x^{11}-1$	$[0.1,\ 1]$
4	x^3+1	$[-1.8,\ 0]$
5	x^3-2x-5	$[2,\ 3]$
6	$(n=5)$ $2\cdot x\cdot e^{-n}+1-2\cdot e^{-n\cdot x}$	$[0,\ 1]$
7	$(n=10)$ $2\cdot x\cdot e^{-n}+1-2\cdot e^{-n\cdot x}$	$[0,\ 1]$
8	$(n=20)$ $2\cdot x\cdot e^{-n}+1-2\cdot e^{-n\cdot x}$	$[0,\ 1]$
9	$(n=5)$ $\left[1+(1-n)^2\right]\cdot x^2-(1-n\cdot x)^2$	$[0,\ 1]$
10	$(n=10)$ $\left[1+(1-n)^2\right]\cdot x^2-(1-n\cdot x)^2$	$[0,\ 1]$
11	$(n=20)$ $\left[1+(1-n)^2\right]\cdot x^2-(1-n\cdot x)^2$	$[0,\ 1]$
12	$(n=5)$ $x^2-(1-x)^n$	$[0,\ 1]$
13	$(n=10)$ $x^2-(1-x)^n$	$[0,\ 1]$
14	$(n=20)$ $x^2-(1-x)^n$	$[0,\ 1]$
15	$(n=5)$ $\left[1+(1-n)^4\right]\cdot x-(1-n\cdot x)^4$	$[0,\ 1]$
16	$(n=10)$ $\left[1+(1-n)^4\right]\cdot x-(1-n\cdot x)^4$	$[0,\ 1]$
17	$(n=20)$ $\left[1+(1-n)^4\right]\cdot x-(1-n\cdot x)^4$	$[0,\ 1]$
18	$(n=5)$ $e^{-n\cdot x}\cdot(x-1)+x^n$	$[0,\ 1]$
19	$(n=10)$ $\left[1+(1-n)^4\right]\cdot x-(1-n\cdot x)^4$	$[0,\ 1]$
20	$(n=20)$ $\left[1+(1-n)^4\right]\cdot x-(1-n\cdot x)^4$	$[0,\ 1]$
21	$(n=5)$ $x^2+sin(x/n)-1/4$	$[0,\ 1]$
22	$(n=10)$ $x^2+sin(x/n)-1/4$	$[0,\ 1]$
23	$(n=20)$ $x^2+sin(x/n)-1/4$	$[0,\ 1]$

1 - **Initial interval with $\xi \in$ [a,b].**

TABLE III

THE NUMBER OF FUNCTION EVALUATIONS WITH FUNCTION g_N AND ROOT-FIND METHOD ϕ_N.

	ϕ_n					
g_n	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
g_1	8	9	10	8	10	11
g_2	11	13	12	11	9	14
g_3	14	15	14	14	15	18
g_4	9	11	11	9	10	12
g_5	8	9	9	8	9	9
g_6	8	10	10	9	10	11
g_7	9	11	12	9	12	11
g_8	9	12	11	10	11	13
g_9	8	10	9	9	10	11
g_{10}	8	11	9	8	10	10
g_{11}	7	11	9	8	9	9
g_{12}	8	10	9	8	10	10
g_{13}	10	11	10	10	11	11
g_{14}	11	13	12	11	12	14
g_{15}	7	13	8	7	8	7
g_{16}	6	17	6	7	7	7
g_{17}	6	20	6	6	6	6
g_{18}	9	9	9	9	9	9
g_{19}	13	14	12	10	11	12
g_{20}	21	21	17	15	15	12
g_{21}	8	8	10	8	10	11
g_{22}	8	9	10	8	10	11
g_{23}	8	9	10	8	10	11
Σg_n	**214**	**276**	**235**	**210**	**234**	**250**

	ϕ_n					
g_n	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}
g_1	11	10	8	9	11	8
g_2	14	12	12	13	15	11
g_3	18	15	13	16	19	22
g_4	13	11	9	11	13	10
g_5	9	9	8	9	11	7
g_6	14	10	9	11	13	8
g_7	16	11	11	11	14	11
g_8	16	10	11	12	10	11
g_9	11	10	10	11	11	8
g_{10}	11	8	9	10	10	7
g_{11}	10	7	8	9	10	6
g_{12}	11	8	9	10	13	8
g_{13}	13	10	10	12	13	9
g_{14}	14	11	11	11	14	11
g_{15}	8	7	7	11	8	7
g_{16}	7	7	7	13	10	6
g_{17}	6	7	6	16	7	6
g_{18}	9	9	9	9	12	7
g_{19}	12	13	10	13	15	8
g_{20}	12	21	15	17	22	9
g_{21}	11	11	8	10	11	8
g_{22}	11	10	9	10	11	8
g_{23}	11	10	9	10	11	8
Σg_n	**268**	**237**	**218**	**265**	**284**	**204**

among other things, if the function $f(x)$ can be computed with absolute error less than the x when x approaches of ξ. For multiple roots convergence is low (only linear). The factors of reduction Φ_7, Φ_6, and Φ_{10} supplies the smallest numbers of iterations, respectively, and are better than the Bisection algorithm. The application of Bisection algorithm uses $n=350$ iterates, deterministically, to reach $<10^{-15}$ of tolerance on diameters $\{b_n-a_n\}$ of isolated single/multiple root when we have $\{b_0-a_0\}=1$ initial.

V. CONCLUSION

We presented a class of regula falsi methods for finding simple zeros of nonlinear equations. In this paper, we have a different reduction factor Φ_n for of each member of family of

TABLE IV
EXAMPLE FUNCTIONS FOR NUMERICAL COMPARISONS OF ROOT-FIND METHODS

n	$f_n(x)$	[a,b][1]
1	$(x+2)\cdot(x+1)\cdot(x-3)^3$	[2.5, 3.5]
2	$(x-4)^5\cdot ln(x)$	[3.5, 4.5]
3	$(sin(x)-x/4)^3$	[2, 3]
4	$(81-p*(108-p*(54-p*(12-p))))*sign(p-3)$ $where\ p=x+1.11111$	[1, 2]
5	$sin((x-7.143)^3)$	[7, 8]
6	$exp((x-3)^5)-1$	[2.5, 3.5]
7	$exp((x-3)^5)-exp(x-1)$	[4, 5]

1 - **Initial interval with $\xi\in$ [a,b].**

TABLE V
THE NUMBER EVALUATIONS OF FUNCTION $\mathbf{F_N}$ WITH THE ROOT-FIND METHOD USING ϕ_N.

	ϕ_n					
f_n	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
f_1	149	96	81	101	63	41
f_2	272	186	149	179	114	82
f_3	150	96	81	101	64	42
f_4	52	33	32	35	18	21
f_5	150	97	80	100	65	41
f_6	45	38	28	27	20	16
f_7	50	50	40	31	32	32
Σf_n	**868**	**596**	**491**	**574**	**376**	**275**

	ϕ_n					
f_n	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}
f_1	43	95	101	50	70	113
f_2	82	185	179	96	85	194
f_3	42	95	102	50	74	115
f_4	17	36	37	19	28	35
f_5	42	98	103	50	68	113
f_6	16	38	28	20	27	22
f_7	31	51	31	37	55	57
Σf_n	**273**	**598**	**581**	**322**	**407**	**649**

methods. The iterations $\{x_n\}$ of the methods have a superlinear convergence for finding simple zeros of nonlinear equations, and the sequence of diameters $\{b_n-a_n\}$ also. In case study 1, numerical experiments show that the new method, with reduction factor Φ_4, is effective and comparable to well-known methods, such as the classical Pegasus, with reduction factor Φ_1 , and Illinois, with reduction factor Φ_8, methods. In case study 2, numerical experiments show that the new methods, with reduction factors Φ_7, Φ_6, and Φ_{10}, are effective and better than the classical Bisection method for finding multiple zeros of nonlinear equations. Asymptotic convergence studies for the new algorithms can be conducted. In each case, we can follow the same type of analysis used to Pegasus, Illinois and Anderson & Björk methods.

APPENDIX

Definition 1: **Multiplicity of a zero of a function**

Let $f:[a,\ b]\in\mathbb{R}\longrightarrow\mathbb{R}$, and let $\xi\in[a,\ b]$ be a zero of f, i.e. a point such that $f(\xi)=0$. The point ξ is said a *zero of multiplicity k* of f if there exist a real number $\mathscr{L}$ such that

$$\lim_{x\to\xi}\frac{|f(x)|}{|x-\xi|^k}=\mathscr{L}$$

Examples from Table IV:

1) $f_1(x)=(x+2)\cdot(x+1)\cdot(x-3)^3$

   $\lim_{x\to 3}\frac{|(x+2)\cdot(x+1)\cdot(x-3)^3|}{|(x-3)|^3}=20$, it follows that 3 is a zero of multiplicity 3.
2) $f_2(x)=(x-4)^5\cdot ln(x)$

   $\lim_{x\to 4}\frac{|(x-4)^5\cdot ln(x)|}{|(x-4)|^5}=ln(4)$, it follows that 4 is a zero of multiplicity 5.
3) $f_5(x)=sin[(x-7.143)^3]$

   $\lim_{x\to 7.143}\frac{|sin[(x-7.143)^3]|}{|(x-7.143)|^3}=1$, it follows that 7.143 is a zero of multiplicity 3.
4) $f_7(x)=exp((x-3)^5)-1$

   $\lim_{x\to 3}\frac{|exp((x-3)^5)-1|}{|(x-3)|^5}=1$, it follows that 3 is a zero of multiplicity 5.

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